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Pareto Set Analysis: Local Measures of Objective Coupling in Multi-objective Design Optimization

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1. Abstract
Multi-objective optimization focuses on the explicit trade-offs between competing criteria. A particular case is the study of combined optimal design and optimal control, or co-design, of smart artifacts where the artifact design and controller design objectives often compete. In the system-level co-design problem, the objective is usually the weighted sum of these two objectives. The preferred practice is to solve co-design problems in a sequential manner: design first, control next. The success of this approach depends on the form of coupling between the two subproblems. In this paper, the coupling vector derived for a system problem with unidirectional coupling is shown to be related to the alignment of competing objectives, as measured by the polar cone of objective gradients, in the bi-objective programming formulation. Further, it is shown how the measure of constraint decoupling can be normalized when the system problem is considered as a bi-objective one. Systematically studying changes to coupling and alignment measures due to changes to the multi-objective formulation can yield deeper insights into the system-level design problem. Two examples illustrate these results.

2. Keywords: Pareto set, sensitivity analysis, co-design, objective coupling, design optimization

3. Introduction
The design of modern smart products requires concurrent optimization of the artifact design and its controller. This so-called co-design problem [1, 2] is often performed in a sequential manner for reasons of convenience and tradition: design the artifact first, and then design its controller. In general, such a strategy will yield non-optimal solutions, compared with a simultaneous or all-in-one optimization of the combined system [1, 2], particularly when bidirectional coupling exists between the two subproblems, for example, when each of the two objectives depends on some variables and parameters of the other subproblem [2]. However, there exists a large class of problems where coupling is unidirectional, for example, the artifact criterion $f_1(\mathbf{x}_1)$ depends only on the artifact design variables $\mathbf{x}_1$ while the control criterion $f_2(\mathbf{x}_1, \mathbf{x}_2)$ depends on both the artifact variables and the controller design variables $\mathbf{x}_2$, so that the system objective becomes: $F = w_1 f_1(\mathbf{x}_1) + w_2 f_2(\mathbf{x}_1, \mathbf{x}_2)$, where w_1, w_2 are weights. An example of such a formulation is a linear positioning device where the artifact objective is steady-state displacement and the controller objective is settling time. Such a partitioning is inherent when the artifact criterion is independent of the controller variables as measured by the partial gradients of the objective and constraint functions with respect to the controller variables. Partitioning artifact and controller variables may be desirable for practical purposes in cases where the effect of the controller variables on the artifact criterion is deemed small enough, or where the analytical or computational means are not available to treat artifact and control variables simultaneously for the controller objective. One strategy for the latter case above is to solve the system-level problem as a nested optimization one [2, 3], where the system solution is found with respect to $\mathbf{x}_1$, with the optimal $\mathbf{x}_2$ computed as a function of $\mathbf{x}_1$ by solving the "inner" optimal controller problem first [1, 3]. This nested problem formulation is distinguished from the simultaneous one using the notation $F^n = w_1 f_1(\mathbf{x}_1) + w_2 f_c^n(\mathbf{x}_2^*(\mathbf{x}_1))$.

Viewing the co-design problem as a bi-objective Pareto formulation without scalarization and weights, we can examine how much the two objectives compete or are aligned [4]. Intuitively, it would appear that objective alignment must relate to objective coupling. Quantifying this relationship will provide deeper insights in the nature of both the alignment and coupling concepts, and their implications for understanding coupled multi-objective problems. In what follows, we show how a measure of objective alignment (the polar cone of objective gradients) is related to the coupling vector derived for a problem with unidirectional coupling, and how the measure of constraint decoupling can be normalized when the system design problem is considered as a bi-objective problem. These measures help to understand how the Pareto set is affected by changes to the system design problem formulation.

Section 4 examines the relationship between objective alignment and objective coupling. Section 5 uses a simple numerical example to illustrate the terms introduced in Section 4. Section 6 presents a system design example that illustrates the relationship between the measures of objective alignment, coupling, and the Pareto set.

4. Objective Coupling and Objective Alignment

Multi-objective programming typically focuses on finding Pareto points and defining the preference structure for selecting one point among many [5]. A Pareto optimization problem is stated as:

$$\min_{\mathbf{x}} \quad \mathbf{f}(\mathbf{x};\mathbf{p})|\ \mathbf{h}(\mathbf{x};\mathbf{p}) = \mathbf{0},\ \mathbf{g}(\mathbf{x};\mathbf{p}) \leq \mathbf{0},\ \mathbf{x} \in \mathcal{X} \tag{1}$$

Here $\mathbf{f}(\mathbf{x};\mathbf{p})$ is a vector of criteria of interest f_i, $i = 1,\ldots,n$. The set of variable values $\mathbf{x}$ that satisfy all constraints is the feasible (design) domain, $\mathcal{X}$. The set of parameters $\mathbf{p}$ take on fixed values. The set of all vectors $\mathbf{f}$ mapped from the feasible domain is the attainable set $\mathcal{A} = \{\mathbf{f}(\mathbf{x};\mathbf{p})|\mathbf{x} \in \mathcal{X}\}$. A point in $\mathcal{A}$, $\mathbf{f}(\mathbf{x}^*)$, is said to be non-dominated or Pareto optimal, if there exist no $\mathbf{f}(\mathbf{x};\mathbf{p})$ such that $\mathbf{f}(\mathbf{x};\mathbf{p}) \leq \mathbf{f}(\mathbf{x}^*;\mathbf{p})$ and $f_i(\mathbf{x};\mathbf{p}) < f_i(\mathbf{x}^*;\mathbf{p})$ for at least one i. Ideal values f_i° are the optimal criterion values obtained optimizing one criterion at a time. The ideal or utopia point is the vector of ideal values for all criteria, $\mathbf{f}^\circ = [f_1^\circ, f_2^\circ]'$.

Several researchers have applied the concept of objective function gradient differences in order to compare solutions [6, 7, 8]. Lootsma examined how the Pareto frontier relates to sensitivity in the objective functions [9]. Additionally, analogies to postoptimal analysis in single objective problems have been proposed, particularly for vector objective linear programming [10, 11]. Others have also discussed the idea of comparing different Pareto sets using the concept of a meta-Pareto set, which includes all non-dominated criteria vectors selected from the union of all the individual Pareto sets under consideration [12, 13]. We adopt the polar cone of the negative gradients as our measure of objective alignment, and we will consider how this measure changes, and the attendant implications for the Pareto set, with changes in the problem formulation.

The decision space can be partitioned into three disjoint sets with respect to a feasible point $\mathbf{x}$: Points $[\mathbf{x}_1,\ldots,\mathbf{x}_n]^\top \in \mathbf{R}^n$ that are superior, $\mathbf{Q}^<(\mathbf{x})$; points that are equal or inferior, $\mathbf{Q}^\geq(\mathbf{x})$; and points that cannot be compared, $\mathbf{Q}^\sim(\mathbf{x})$. The set $\mathbf{Q}^<(\mathbf{x})$ is equivalent to the interior of the polar cone of the negative objective gradients

$$\mathbf{Q}^<(\mathbf{x}) = \{\mathbf{k}| -\mathbf{k}^\top \nabla f^i > \mathbf{0}; i = 1,2\} \tag{2}$$

where $\mathbf{k}$ is an n-dimensional vector with origin at $\mathbf{x}$ [14, 4]. The angle between the boundaries of the polar cone can then be taken as a measure of objective function alignment at a particular $\mathbf{x}$. A polar cone angle of π corresponds to the case where the gradients of both objectives at $\mathbf{x}$ are parallel. The polar cone angle collapses to 0 when objective gradients are parallel with reversed signs.

The interdependence of the multiple objectives for a given system is critical to its design [15, 16, 17]. The complete co-design problem with unidirectional coupling is formulated as [3]

$$\begin{aligned} \min_{\mathbf{x}_1,\mathbf{x}_2} \quad & w_1 f_1(\mathbf{x}_1) + w_2 f_2(\mathbf{x}_1,\mathbf{x}_2) \\ \text{subject to:} \quad & \mathbf{h}_1(\mathbf{x}_1) = \mathbf{0};\ \mathbf{h}_2(\mathbf{x}_1,\mathbf{x}_2) = \mathbf{0} \\ & \mathbf{g}_1(\mathbf{x}_1) \leq \mathbf{0};\ \mathbf{g}_2(\mathbf{x}_1,\mathbf{x}_2) \leq \mathbf{0} \end{aligned} \tag{3}$$

where $f_1(\mathbf{x}_1)$ is the artifact objective function, $f_2(\mathbf{x}_1,\mathbf{x}_2)$ is the controller objective function, $\mathbf{x}_1$ is the vector of artifact design variables, $\mathbf{x}_2$ is the vector of controller design variables, $\mathbf{h}$ are the system equality constraints, and $\mathbf{g}$ are the system inequality constraints, and w_1 and w_2 are the weights associated with the objective functions f_1 and f_2, respectively.

4.1. Definitions

We adopt several terms to aid in explaining changes in the Pareto set. First, we define terms related to the unidirectional coupled system problem. Next, we define terms related to the bi-objective problem. Then, we define terms related to both problems.

Consider the nested system design problem $F^n = w_1 f_1(\mathbf{x}_1) + w_2 f_2^n(\mathbf{x}_2^*(\mathbf{x}_1))$, where the asterisk denotes that the optimal values for $\mathbf{x}_2$ have been found with respect to $\mathbf{x}_1$. The coupling vector $\mathbf{\Gamma_v}$ [3] is derived from the Karush-Kuhn-Tucker (KKT) optimality conditions for the weighted-sum objective describing the system design problem, where $W = w_2 | w_2 > 0, w_1 = 1$. The inner term is the gradient $\nabla f_2^N(\mathbf{x}_1)$. $\mathbf{\Gamma_v}$ is assumed to be a row vector.

$$\mathbf{\Gamma_v} = W \left(\frac{\partial f_2}{\partial \mathbf{x}_1} + \frac{\partial f_2}{\partial \mathbf{x}_2^*} \frac{\partial \mathbf{x}_2^*}{\partial \mathbf{x}_1} \right) = W \nabla f_2^n(\mathbf{x}_1) \tag{4}$$

Objective decoupling occurs when the inner term of $\mathbf{\Gamma_v}$ vanishes. In this case the solution to the single-objective problem, min f_1, will also be the solution to the weighted-sum system objective problem [3].

Constraint decoupling occurs when there is a range of values for W for which a given $\mathbf{x}^*$ is the system optimal solution. This behavior occurs when the gradients of the active constraints at the system optimal solution can form convex combinations equal to the system objective gradient for a range of objective gradient directions, controlled by W.

For a bi-objective problem, two objectives are said to be aligned at a particular design point $\mathbf{x}$ if the angle between the objective gradients is 0, or equivalently if the angle described by the polar cone $\theta^<$ of the two negative objective gradients is π. The polar cone has an appealing geometric interpretation in that the larger $\theta^<$ the greater the region of simultaneously improving directions, or the greater the objective alignment. From the definition of polar cone in Equation 2, the polar cone angle is

$$\theta^< = \arccos \left(\frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{|\mathbf{k}_1||\mathbf{k}_2|} \right), \ \{\mathbf{k}_1 | -\mathbf{k}_1 \nabla f_1(\mathbf{x}) = 0; \ \mathbf{k}_2 | -\mathbf{k}_2 \nabla f_2(\mathbf{x}) = 0\}. \tag{5}$$

A convenient way to identify the appropriate $\mathbf{k}_1$ and $\mathbf{k}_2$ for a problem with two design variables is to recognize that $\mathbf{k}_1$ should be orthogonal to $-\nabla f_1$ and in the plane defined by f_1 and f_2. We preserve the polar cone measure for the n-dimensional case but calculate it directly from the objective gradients: $\theta^< = \pi - \alpha$, where $\alpha = \arccos((\nabla f_1 \cdot \nabla f_2)/(|\nabla f_1||\nabla f_2|))$.

Two objective criteria are said to be coincident when the single-objective minimizers of the design variables shared between the objectives are equal. Namely, there exists some vector $\mathbf{x}^* | \ f_1(\mathbf{x}^*) = f_1^\circ, f_2(\mathbf{x}^*) = f_2^\circ$. A relative measure of coincidence to compare two Pareto sets is the L_2 norm of the design variables between objective ideal points $||(\mathbf{x}^{f_1^\circ} - \mathbf{x}^{f_2^\circ})||_2$.

One Pareto set is said to dominate another Pareto set when each member of the dominated Pareto set belongs to $\mathbf{Q}^{\geq}(\mathbf{x})$ for at least one member of the dominating Pareto set.

Two objectives are said to be independent at a particular design point $\mathbf{x}$ when $\mathbf{\Gamma_v} = \mathbf{0}$ or when $\theta^<$ is undefined.

The coupling vector $\mathbf{\Gamma_v}$ is related to the slope of the Pareto frontier of the bi-objective problem [18], where at a given Pareto-efficient point $\mathbf{x}^*$

$$\frac{df_2^*}{df_1^*} = \frac{1}{W} \mathbf{\Gamma_v} \frac{d\mathbf{x}_1}{df_1^*} = \nabla f_2^{n\top} \cdot (1/\nabla f_1). \tag{6}$$

4.2. Quantification of Alignment and Objective Coupling

We begin with the necessary conditions for an efficient point to a bi-objective minimization problem [19] as in Equation 1:

$$\eta_1 \nabla f_1(\mathbf{x}^*) + \eta_2 \nabla f_2(\mathbf{x}^*) + \lambda^\top \nabla \mathbf{h} + \mu^\top \nabla \mathbf{g}(\mathbf{x}^*) = \mathbf{0}; \mu \geq \mathbf{0}; \lambda \neq \mathbf{0}; \mu^\top \mathbf{g}(\mathbf{x}^*) = \mathbf{0}; \mathbf{h} = \mathbf{0} \tag{7}$$

Comparing Equation 7 to what would be the first-order optimality conditions for Equation 3 we see that the co-design problem is a special case of the bi-objective problem where the weighting factors were chosen a priori. In previous work on co-design coupling, emphasis has been placed on comparing f_1 to the weighted system objective $w_1 f_1 + w_2 f_2$ rather than comparing f_1 and f_2 directly. However, the coupling vector $\mathbf{\Gamma_v}$ is difficult to interpret because it is directly proportional to the subjective weighting value W and the units of measurement for the objective function. Comparing the two objectives directly frees the designer from implying a scale W, or "exhange rate", between objectives before studying the attainable set. Objective alignment is a function of gradient direction only (not magnitude). It is still possible that the gradient direction is affected by the scale of the controller variables $\mathbf{x}_2$ since they do not appear in the artifact objective function.

Objective alignment can be calculated for a unidirectional coupled problem following the definition of alignment above. This is a straightforward process if the whole gradients are available: $\nabla f_1(\mathbf{x}_1), \nabla f_2(\mathbf{x}_1, \mathbf{x}_2)$.

However, it can be challenging to formulate or compute the whole gradients when the controller objective is formulated as a nested problem with gradient $\nabla f_2^n = \frac{\partial f_2}{\partial \mathbf{x}_1} + \frac{\partial f_2}{\partial \mathbf{x}_2^*}\frac{\partial \mathbf{x}_2^*}{\partial \mathbf{x}_1}$. For example, assume $f_2^n(\mathbf{x}_1)$ is a black-box simulation. We can then compute $\nabla f_2^n(\mathbf{x}_1)$ and observe $\partial \mathbf{x}_2^* / \partial \mathbf{x}_1$. However, we require $\nabla f_2(\mathbf{x}_1, \mathbf{x}_2)$. If we assume we can compute $\nabla f_2(\mathbf{x}_2)$ analytically or by evaluating the conventional controls problem $f_2(\mathbf{x_2})$, then we can back out the missing component: $\partial f_2 / \partial \mathbf{x}_1 = \nabla f_2^n - \frac{\partial f_2}{\partial \mathbf{x}_2}\frac{\partial \mathbf{x}_2}{\partial \mathbf{x}_1}$.

4.3. Normalized Constraint Decoupling

Returning to the geometric interpretation of the weighted-sum objective in the design variable space, the sum of any two vectors with positive weighting factors $((1-w)f_1 + wf_2 | \ w \geq 0)$ will be a new vector that lies between the two original vectors assuming the same origin. The necessary conditions for the optimal system design problem imply that the weighted-sum-objective vector can be formed by a convex combination of the gradients of the active constraints. Constraint decoupling requires that, at a given Pareto point for a system design problem with weight w, the span of the convex combination of satisfied constraints (including degenerate constraints) will similarly satisfy the necessary conditions for optimality for a system design problem with some other weighting factor $\omega \neq w$.

The constraint decoupling ratio ϕ can be calculated at an ideal point (f_1°) by first calculating the angle between the two single-objective gradients $\nabla f_1, \nabla f_2$. We can then find the limiting weighting value w^*, and compute the angle between the weighted-sum system-objective gradient and ∇f_1. Assuming the system design objective is a convex combination of the single objectives, the limiting weighting value can be found by solving the following problem where $\nabla f_1, \nabla f_2, \nabla \mathbf{g}, \nabla \mathbf{h}$ have been evaluated at $(\mathbf{x}_1, \mathbf{x}_2)_{f_1^\circ}$.

$$\begin{aligned} \min_{w,\boldsymbol{\beta}} \quad & -w \\ \text{subject to:} \quad & (1-w)\nabla f_1 + w\nabla f_2 + \boldsymbol{\beta}^\top \nabla \mathbf{g} + \boldsymbol{\lambda}^\top \nabla \mathbf{h} = \mathbf{0} \\ & -\boldsymbol{\beta} \leq \mathbf{0}; \boldsymbol{\lambda} \neq \mathbf{0} \end{aligned} \tag{8}$$

The ratio ϕ evaluated at an ideal point is then the ratio of the angle between the maximum weighted-sum objective gradient with the same optimal solution as the ideal point and the single-objective gradient, and the angle between the two single-objective gradient vectors:

$$\phi = \arccos\left(\frac{\nabla f_1 \cdot ((1-w^*)\nabla f_1 + w^*\nabla f_2)}{|\nabla f_1| \, |((1-w^*)\nabla f_1 + w^*\nabla f_2)\,|}\right) / \arccos\left(\frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1||\nabla f_2|}\right) \tag{9}$$

The amount of constraint decoupling, or the range of weighting values for which the constraint decoupling conditions hold, will change with the objective scaling. However, ϕ, based on the gradient directions will take a value between 0 and 1 and will not change with objective scaling. Figure 2(c) illustrates this case where the dashed line shows the limiting gradient direction for which the system design problem will have the same solution as the single-objective problem f_1. A normalized measure can be defined for any Pareto point $\mathbf{x}^*$ by replacing ∇f_1 in Equation 8 with $(1-\upsilon)\nabla f_1 + \upsilon \nabla f_2$, where υ is the minimum weighting value for which $\mathbf{x}^*$ is the system design problem solution.

Given a fixed set of constraints, increasing objective alignment will result in a higher ϕ. Given a fixed set of objectives, decreasing satisfied-constraint alignment will result in higher ϕ.

Figure 1 summarizes the steps described above into an algorithm for systematically analyzing the Pareto set for objective alignment and coupling. The "compare results" step of the algorithm is illustrated in the next sections.

5. Pareto Set Analysis

The design of the solution set (rather than a single-point design) is important in many design scenarios. These scenarios share a characteristic that design decisions are not all made simultaneously, but some may be made before others (configuration design), some decisions may be more flexible then others, or be repeated at a higher frequency (dynamic controls, product platforming, design for adjustability), and some decisions (or exogeneities) may be uncertain (robust design, product development investment planning, regulatory policy). In the general case, systems characterized by multiple objectives will exhibit a tradeoff relationship between improvements for both objectives. Considering how the Pareto set changes with changes in the problem formulation can facilatate design of the attainable set in addition to illustrating the tradeoffs between specific solutions.

Changes to the mathematical structure and input parameter values of a bi-objective programming problem can lead to changes in the shape of the attainable set and its Pareto boundary. We illustrate the

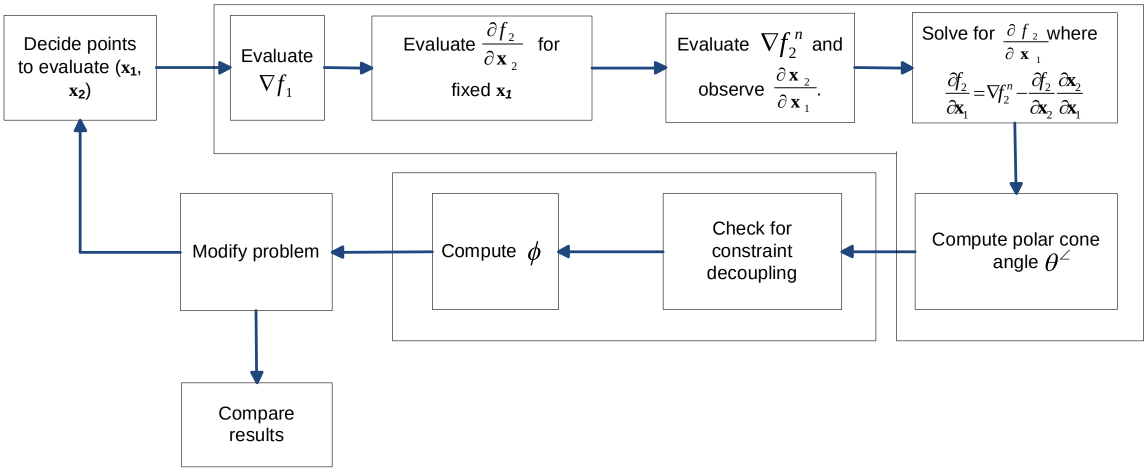


Figure 1: Algorithm for evaluating objective alignment and coupling measures for multiple problem formulations

link between the terms described in Section 4 and outcomes of the Pareto set using a two-dimensional nonlinear programming example, and a linear-positioning device design.

5.1. Problem specification and identification

The task of the designer abstracted to a mathematical decision-making problem is to specify the functional forms of the objective and constraint functions (referred to as system specification in the dynamic systems terminology), then partition model elements between parameters and variables, specify parameter values, and find efficient values for design variables (referred to as system identification in the dynamic systems terminology). We classify changes to a system design problem formulation (summarized in Table 1) according to this definition of system specification and identification. Each of these decisions may affect the Pareto set. For example, changing parameter values is equivalent to a traditional parametric study and fits in system identification. The examples listed in Table 1 reflect changes to the example problem specified in Equation 10.

Table 1: Classification of System Design Model Changes

System Modeling Stage	Change	Example
Specification	Objective functional form	-
	Constraint functional form	-
	Add constraint	$x_2 - 5 + 2x_1 \leq 0$
	Remove constraint	-
Identification	Repartition parameters	-
	Parameter values	$p = 5$

5.2. Example

We now demonstrate examples of problem formulation changes and observe the corresponding changes to the Pareto set for a two-variable nonlinear programming problem modified from problem 10 in [20]:

$$\begin{aligned} &\min\ [f_1 = 0.5x_1^2 - 7x_1,\ f_2 = x_2^2 - x_1x_2 - px_2]^\top \\ &\text{subject to: } x_2^2 + 4x_1^2 - 25 \leq 0,\ p = 7 \end{aligned} \quad (10)$$

The problem is illustrated graphically in Figure 2. Case (a) shows the unmodified problem, case (b) shows the case where $p = 5$, and case (c) includes the additional constraint $x_2 - 5 + 2x_1 \leq 0$. The dashed line in case (c) indicates the degree of constraint decoupling for the problem.

We compute the measures defined in Section 4 and report the values in Table 2. The results have been categorized by their reflection on the Pareto set. Performance refers to the placement of the Pareto

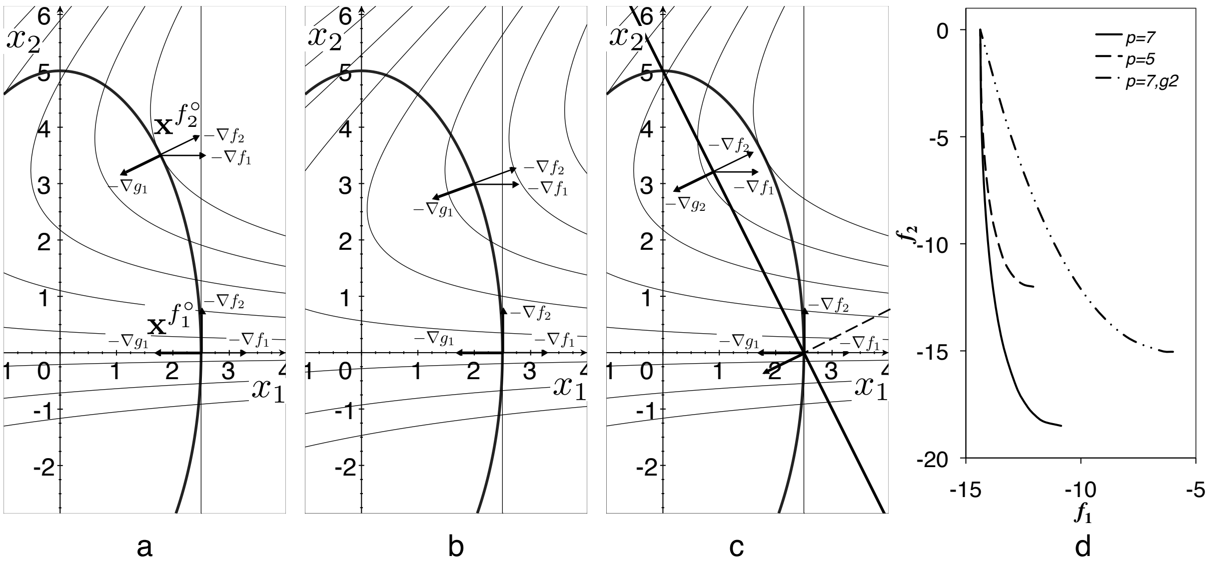


Figure 2: Graphical representation of problem given in Equation 10 and modifications from Table 1. Panel (a) shows the unmodified problem. Panel (b) shows the case where $p = 5$. Panel (c) shows the case where the constraint $x_2 - 5 + 2x_1 \leq 0$ has been added

set in the objective space. When one Pareto set dominates another Pareto set it is said to have improved performance. Sensitivity refers to the the slope of the Pareto frontier. Evaluated locally, sensitivity is defined as df_2^*/df_1^*. Evaluated over the entire Pareto frontier it represents the cost in one objective to achieve the ideal value for the other objective. Parity refers to the similarity in the decision to be made in order to minimize each objective f_1 and f_2 singly. One measure of parity is the measure of coincidence between ideal values $(||(\mathbf{x}^{f_1^\circ} - \mathbf{x}^{f_2^\circ})||_2)$. Complete parity requires that the decision maker chooses the identical decision in order to minimize both objectives, in other words, the objectives are coincident. We use the polar cone angle of the negative objective gradients $\theta^<$ evaluated at an ideal point $\mathbf{x}^{f_1^\circ}$ as an alternative measure of parity given that objectives with a polar cone angle $= \pi$ will be coincident. A degenerate case of complete parity is when the objectives are independent.

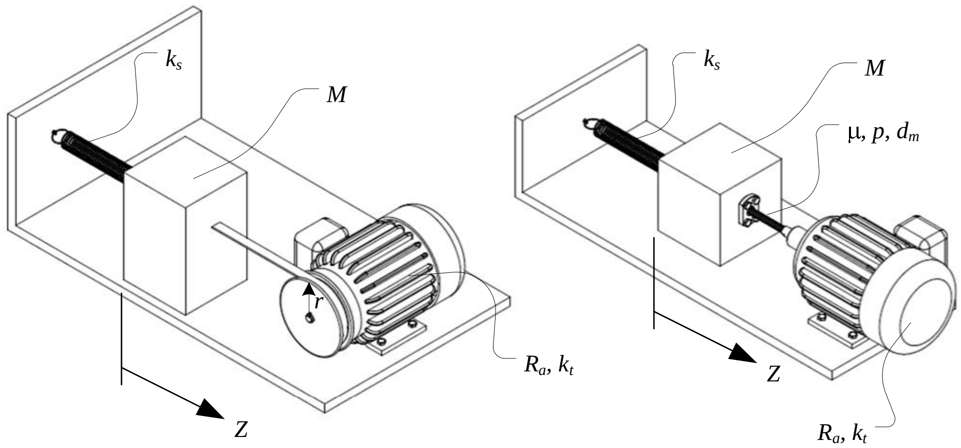
Examining Table 2, case (a) dominates the other cases and so is superior in the performance criterion. However, if parity is important, then case (b) would be superior. For reducing sensitivity of f_2 with respect to f_1 case (c) would be superior.

Table 2: Pareto Set Analysis Results

Criterion		Case	a	b	c
Performance	Dominance		$a \geqslant b$, $a \geqslant c$	-	-
Sensitivity	Local	f_1°	und.	und.	und.
		f_2°	0	0	0
	Global (f_2/f_1)		-5.26	-5.21	-1.80
	$\mathbf{\Gamma_v}$	f_1°	und.	und.	und.
		f_2°	0	0	0
	ϕ		0	0	0.29
Parity	Coincidence		3.59	3.00	3.54
	Polar cone angle	f_1°	$\pi/2$	$\pi/2$	$\pi/2$
		f_2°	2.68	2.79	2.68

6. System Design Example

We demonstrate our described Pareto analysis for a simplified design and controls problem involving a positioning gantry. Changes in problem formulations could arise through changes in the mechanism selected or the choice of design variables. We consider two system topologies. In the first topology, shown in Figure 3a, a mass M is connected to a fixed surface by a linear spring with constant k_s. A flexible belt connects the mass and a pulley with radius r, which is mounted on a DC motor with armature resistance R_a and motor constant k_t. The displacement of the mass from its original position is Z. In the second topology, shown in Figure 3b, the belt and pulley are replaced by a power screw with diameter d_m, coefficient of friction μ, and pitch length p. A state-feedback controller with a precompensator is applied to the system, as shown in Figure 4.



(a) Positioning Gantry with Drive Belt and Pulley (b) Positioning Gantry with Power Screw

Figure 3: Linear Positioning System

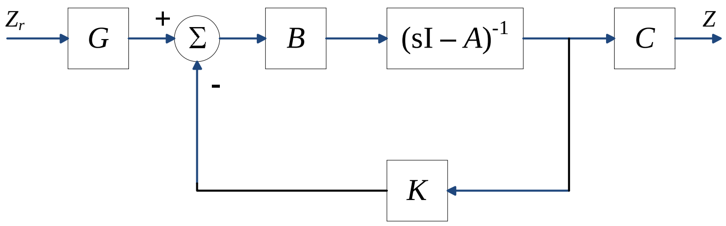


Figure 4: Control Architecture for Linear Positioning System

The optimization problem for both topologies is formulated as

$$\min_{M,k_s,K_1,K_2} F = w_1 f_1 + w_2 f_2 \quad (11)$$

$$f_a = -Z_f(M, k_s) \quad (12)$$

$$f_c = t_s(M, k_s, K_1, K_2) \quad (13)$$

subject to

$$g_1(M, k_s) = M - c_1 k_s^{2/3} \leq 0 \quad (14)$$

$$g_2(M, k_s, K_1, K_2) = M_p - M_{p,max} \leq 0 \quad (15)$$

$$g_3(M, k_s, K_1, K_2) = E - E_{max} \leq 0 \quad (16)$$

where c_1 is a constant based on the material strength of the spring, M_p is the overshoot in the position response, and E is the control effort over a set interval of time, calculated as $E = \int_0^{t_f} (V(t))^2 \, dt$.

Parameters used in the optimization are given in Table 3. The Pareto frontiers for the two topologies are given in Figure 5, and a comparison is given in Table 4. For some measures, there is no difference between the two topologies. For example, neither exhibits constraint decoupling, and in both cases the polar cone angle $\theta^{<} = \pi/2$ at all points along the Pareto set. There are some differences, however, which can be used to judge which topology might be desirable. It can be seen that the Pareto frontier

Table 3: Parameters for Gantry Optimization

Parameter	r	k_t	R_a	c_1	V_f	μ	d_m	p
Value	2.5 cm	10.0 N-m/A	2.0 kΩ	1.0	10 V	0.06	50 mm	10 mm

corresponding to the first topology, with the belt drive, is dominant. Sensitivity analysis indicates that the belt drive topology is less sensitive, both for the particular points chosen and for the global measure. This might also indicate that the belt drive is a better choice, if low sensitivity is considered desirable in the problem. The coupling vector $\boldsymbol{\Gamma_v}$, however, has a smaller magnitude for the power screw topology for a number of points on the curve, including the points labeled as 'A' on each Pareto frontier. This might argue in favor of the power screw topology, if reducing the amount of coupling is of importance. The power screw is also favored by an analysis of coincidence. The choice between these options will therefore depend on a full understanding of the design problem and what is considered to be important by the designer. Different circumstances will dictate the relevant Pareto set characteristics.

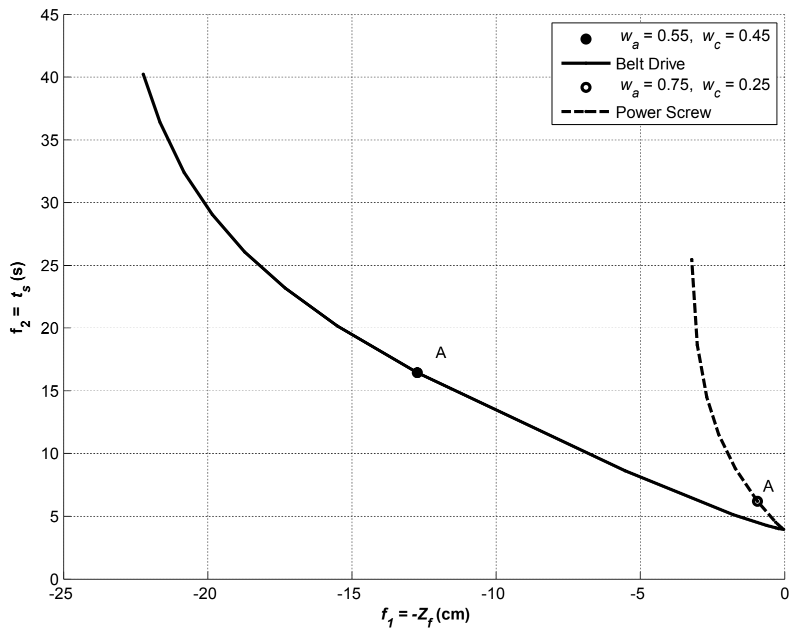


Figure 5: Comparison of Pareto Frontiers for Linear Positioning Device

7. Conclusions

We have shown how the particular case of combined optimal design and optimal control, or co-design, can be represented as a system design problem or alternatively as a bi-objective programming problem with an artifact objective and a controller objective. The coupling vector derived for a system problem with unidirectional coupling was shown to be related to the alignment of competing objectives, as measured by the polar cone of objective gradients, in the bi-objective programming formulation. We also showed how the measure of constraint decoupling can be normalized when the system problem is considered as a bi-objective one.

Systematically studying changes to coupling and alignment measures due to changes to the multi-objective formulation can yield deeper insights into the system-level design problem. We illustrated this approach with a simple nonlinear programming example and a positioning gantry co-design problem. The approach involves selecting several Pareto efficient points for analysis. At a minimum, the two ideal points should be selected. These points can be evaluated for each change in problem formulation such as a change in the functional form of the objective function or a repartitioning of variables and parameters. Numerical measures were then presented that describe each Pareto set in terms of its performance or

Table 4: Pareto Set Analysis Results for Positioning Gantry

Criterion			Belt Drive	Power Screw
Performance	Dominance		$a \geqslant b$	-
Sensitivity	Local	f_1°	und.	und.
		A	-1.2	-3.1
		f_2°	0	0
	Global (f_2/f_1)		-1.64	-6.67
	$\boldsymbol{\Gamma_v}$	f_1°	$\begin{bmatrix} 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \end{bmatrix}$
		A	$\begin{bmatrix} -6.95 & -8.01 \end{bmatrix}$	$\begin{bmatrix} -0.277 & -0.234 \end{bmatrix}$
		f_2°	und.	und.
	ϕ		0	0
Parity	Coincidence		262	147
	Polar Cone Angle		$\pi/2$ (all points)	$\pi/2$ (all points)

dominance over other problem formulations, sensitivity of one objective to changes in the other objective, and the parity faced by the decision maker when considering the difference between the single-objective solutions.

The desirability of a given Pareto set, or problem formulation, over another should be dictated by the design context. There are numerous design contexts where the designer is concerned with the attainable solution set and not only a point design solution. Such problems include the nested co-design problem described here and a wide variety of problems where the "control," or partitioned variable design decisions, may be made asynchronously to the other design decisions. The performance, sensitivity, and parity attributes of a particular design problem represent a multi-objective problem of their own.

It should be noted that objective alignment on the Pareto set as measured by the polar cone of the objective gradients is different from the general notion of objective alignment that is perhaps best characterized by the measure of coincidence. In fact high polar cone angles along the Pareto frontier could be an indication of an undesirable problem formulation in some design scenarios. For example, in an unconstrained bi-objective problem, by definition the polar cone angle will be 0 at all points along the Pareto set. High polar cone angles along the Pareto set may imply that the feasible design space is far away from both unconstrained objective optima.

Increased parity may similarly result from constraint tightening (either through parameter changes or adding constraints). Both polar cone angle increases and lower coincidence distance may also be achieved through reformulation of the objective functions.

The notion of sensitivity is particularly useful for problems characterized by a primary objective and a secondary objective. For example in a product design problem with a market system objective such as maximize profit, a producer will primarily be concerned with satisfying the profit objective although the producer may also be concerned with other objectives such as environmental impact for strategic or other reasons. In this case it would be valuable to assess the local sensitivity, or incremental cost to the profit objective for decreases in environmental impact. Decreased sensitivity may result from a decrease in the coupling term due to constraint reformulation, objective reformulation, parameter/variable repartitioning, or modification to parameter values.

Plots of Pareto sets such as Figures 2 and 5 are typical and very useful analysis tools for bi-objective problems. However, as computational expense increases, it is not always feasible to generate a suitable graphical representation of the Pareto set. The methods presented in this paper provide a series of concrete steps to make the most out of a small number of analysis runs by connecting the local behavior of a bi-objective problem to the characteristics of the Pareto set. Future work may seek to extend the numerical measures described here to higher-dimension problems where graphical representation is similarly problematic.

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